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On energy and enstrophy exchanges in two-dimensional non-divergent flow

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It is shown that, in two-dimensional non-divergent flow in a bounded region, roughly 70% of triad interactions exchange more energy with longer wavelengths than with shorter wavelengths whilst roughly 40% exchange more enstrophy with longer wavelengths than with shorter wavelengths.

1. Introduction

There is a statement in Fjørtoft's (1953) paper on two-dimensional energy exchange which is in error and which has often been quoted (Kraichnan 1967; Charney 1971; Lilly 1967; Green 1974) to indicate a fundamental difference between two- and three-dimensional turbulence. The idea in error is that a transfer of energy from one wavenumber to a higher one must be accompanied by a transfer of *still more energy* to a lower wavenumber. It is correct that in any exchange energy must simultaneously flow up and down scale, but it is not correct to say that more must flow in one direction or the other in all triad interactions. This misunderstanding probably results from the error in equation (20) of Fjørtoft's paper.

The purpose of this paper is to re-examine this question and estimate what fraction of interactions transfers more energy to low wavenumbers and vice versa. To do this, we shall first consider two-dimensional non-divergent flow in a doubly periodic domain. This case has been used by many workers to perform numerical simulations of two-dimensional turbulence and is more easily treated than flow on a sphere. Subsequently, we shall treat the case of flow on a sphere in analogy with the plane case. It should also be noted that our discussion of the triad interactions is only concerned with the energy exchange and gives no method for determining the sign of the energy flow. To deduce the direction of energy flow further assumptions must be made such as Batchelor's (1953, p. 186) hypothesis that an initially narrow spectrum spreads out about its centroid.

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2. Fjørtoft's theorem

Consider the two-dimensional non-divergent flow of an inviscid fluid on a plane. The evolution of this flow is governed by the vorticity equation

$$\partial(\nabla^2\psi)/\partial t = -\mathbf{V}.\nabla(\nabla^2\psi), \tag{1}$$

where $\mathbf{V} = \mathbf{k}\times\nabla\psi$; $\mathbf{k}$ is a unit vector perpendicular to the plane of fluid flow and ψ is a stream function. If we further confine the flow to be periodic in the plane co-ordinates (x, y), then we may expand ψ in a Fourier series such that

$$\psi(x,y) = \Sigma\psi_{\mathbf{K}}e^{i\mathbf{K}.\mathbf{R}}, \tag{2}$$

where $\mathbf{K} = \mathbf{k}_x\mathbf{i}+\mathbf{k}_y\mathbf{j}$ and $\mathbf{R} = x\mathbf{i}+y\mathbf{j}$. Further, as shown by Lorenz (1960), when (1) is transformed into a set of equations for the evolution of $\psi_{\mathbf{K}}$, its right-hand side takes a form such that energy is exchanged among a triad of components $(\psi_{\mathbf{K}}, \psi_{\mathbf{L}}, \psi_{\mathbf{M}})$ if $\mathbf{K}+\mathbf{L}+\mathbf{M} = 0$ and none of $\mathbf{K}$, $\mathbf{L}$ or $\mathbf{M}$ are collinear. Also, there are two constants of motion of this flow which are thought to be fundamental, namely the energy E and enstrophy V. These two quantities may be expressed as

$$E = \Sigma\,(\mathbf{K}.\mathbf{K})\,\psi_{\mathbf{K}}\psi^*_{\mathbf{K}}, \quad 2V = \Sigma\,(\mathbf{K}.\mathbf{K})^2\,\psi_{\mathbf{K}}\psi^*_{\mathbf{K}}. \tag{3), (4}$$

Consider now the energy exchange among an interacting triad of wavenumbers (K, L, M). If we define $E_K = (\mathbf{K}.\mathbf{K})\,\psi_{\mathbf{K}}\psi_{-\mathbf{K}}$ and consider variations of the energies for this triad in the manner of Fjørtoft (1953), then in any interaction

$$\delta E_K+\delta E_L+\delta E_M = 0, \tag{5}$$

$$(\mathbf{K}.\mathbf{K})\,\delta E_K+(\mathbf{L}.\mathbf{L})\,\delta E_L+(\mathbf{M}.\mathbf{M})\,\delta E_M = 0. \tag{6}$$

Without loss of generality, we may define $\mathbf{K}$, $\mathbf{L}$ and $\mathbf{M}$ such that $|\mathbf{K}| \leqslant |\mathbf{L}| \leqslant |\mathbf{M}|$ and then if we define $X = \delta E_K/E\delta_L$ and $Y = \delta E_M/\delta E_L$, (5) and (6) may be solved to obtain

$$X = \frac{\mathbf{M}.\mathbf{M}-\mathbf{L}.\mathbf{L}}{\mathbf{K}.\mathbf{K}-\mathbf{M}.\mathbf{M}}, \quad Y = \frac{\mathbf{L}.\mathbf{L}-\mathbf{K}.\mathbf{K}}{\mathbf{K}.\mathbf{K}-\mathbf{M}.\mathbf{M}}. \tag{7}$$

From (7) and the ordering assumption, it follows that X and Y are both negative, which shows that in any exchange energy must flow both up and down scale.

Let us now consider the relative magnitude of the energy flows. Let $S = X/Y$; then if $S > 1$ more energy flows to or from longer wavelengths and if $S < 1$ more energy flows to or from shorter wavelengths.

We shall now focus on a particular value of $\mathbf{L}$ and consider all possible interactions consistent with the assumed ordering and ask if S is consistently greater or less than 1. Using the fact that $\mathbf{M} = -(\mathbf{K}+\mathbf{L})$ and (7), we have

$$S = (\mathbf{K}.\mathbf{K}+2\mathbf{L}.\mathbf{K})/(\mathbf{L}.\mathbf{L}-\mathbf{K}.\mathbf{K}). \tag{8}$$

We require that $S \geqslant 0$ (because $|\mathbf{M}| \geqslant |\mathbf{L}|$) and further $|\mathbf{K}|/|\mathbf{L}| \leqslant 1$. Now suppose $\mathbf{L}.\mathbf{K} = 0$ and $|\mathbf{K}| = |\mathbf{L}|\,(1-\epsilon)$, where ϵ is small; then it follows that

$$S \simeq (1-2\epsilon)/2\epsilon > 1.$$

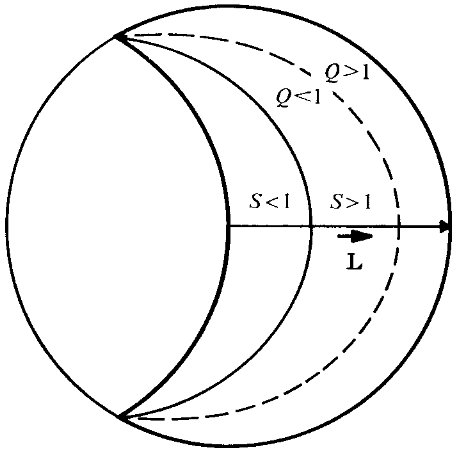

FIGURE 1. Characteristics of energy and enstrophy exchange in plane co-ordinates. **K** vectors may terminate anywhere within the region outlined by the thick line. Those terminating to the left of the other solid line imply $S < 1$; those terminating to the left of the dashed line imply $Q < 1$. The numbers of interactions in each region are proportional to the enclosed areas.

On the other hand, if $\mathbf{L}.\mathbf{K} = 0$ and $|\mathbf{K}| \simeq \epsilon|\mathbf{L}|$ with ϵ small, then

$$S \simeq \epsilon^2 < 1.$$

Thus, we can see that S is not consistently greater or less than 1.

3. The relative magnitudes of energy and enstrophy flow

We have seen that S cannot always be greater or always less than 1. It is of interest, then, to inquire as to the number of interactions which satisfy one condition or the other and their distribution in wavenumber space. To answer this question, we again focus on a particular wave vector **L** and consider all possible interactions with wave vectors **K** and **M** consistent with the ordering $|\mathbf{K}| \leqslant |\mathbf{L}| \leqslant |\mathbf{M}|$. If we define θ as the angle between **L** and **K**, then we have

$$S = \frac{\mathbf{K}.\mathbf{K}+2|\mathbf{L}|\,|\mathbf{K}|\cos\theta}{\mathbf{L}.\mathbf{L}-\mathbf{K}.\mathbf{K}} = \frac{\alpha^2+2\alpha\cos\theta}{1-\alpha^2}, \tag{9}$$

where $\alpha = |\mathbf{K}|/|\mathbf{L}|$.

Consider now a rectangular mesh of points such that the line segments directed from the origin to the points represent the possible wave vectors. Then a circle of radius $|\mathbf{L}|$ will enclose all possible wave vectors **K** consistent with the assumed ordering. Also, since $S \geqslant 0$, $\cos\theta \geqslant -\frac{1}{2}\alpha$ is a further condition imposed by our assumed ordering. In figure 1, we show the region in which a **K** vector may terminate. Further, if $S = 1$, then

$$\alpha_c(\theta) = \tfrac{1}{2}\{-\cos\theta + (\cos^2\theta+2)^{\frac{1}{2}}\} \tag{10}$$

and thus the curve $\alpha_c(\theta)$ divides the area bounded by $\alpha \leqslant 1$ and $\cos\theta \geqslant -\frac{1}{2}\alpha$ into two regions. The region on the left corresponds to $S < 1$, the other $S > 1$.

We can obtain an estimate of the relative number of interactions in these two regions by measuring the relative areas of these regions, since each possible terminating point occupies (on the average) the same area. Since we are interested in the relative number, we may consider that **L** has unit magnitude. Then if

$$A_1 = 2\int_0^1 \int_0^\phi r\,d\theta\,dr \quad (\cos\phi = \tfrac{1}{2}r)$$

and

$$A_2 = 2\int_0^{\frac{2}{3}\pi} \int_{\alpha_c(\theta)}^1 r\,dr\,d\theta,$$

A_2/A_1 is a measure of the fraction of interaction for which $S > 1$. We find that

$$A_1 = \tfrac{1}{3}\pi + \tfrac{1}{2}\surd 3, \quad A_2 = \tfrac{7}{24}\pi + \tfrac{1}{4}\surd 3,$$

so that roughly 70 % of possible interactions have $S > 1$ and 30 % have $S < 1$. These figures were verified by actually counting interactions for a few cases.

As we have seen, about 70 % of nonlinear interactions lead to a larger exchange of energy with lower wavenumbers. The situation is almost reversed in the case of enstrophy flow. For if we define Q as the ratio of enstrophy flow to lower wavenumbers to enstrophy flow to higher wavenumbers, then

$$Q = \frac{\mathbf{K}.\mathbf{K}}{\mathbf{M}.\mathbf{M}} S = \frac{\alpha^2 + 2\alpha\cos\theta}{1-\alpha^2}\,\frac{\alpha^2}{\alpha^2 + 2\alpha\cos\theta + 1}. \tag{11}$$

The curve $Q = 1$ then divides the region in two, as shown by the dashed line in figure 1. We note that the area where $Q < 1$ is now greater than that where $Q > 1$. By numerical quadrature, we find that roughly 60 % of interactions exchange more enstrophy with higher wavenumbers.

4. Extension to spherical geometry

In spherical co-ordinates, the appropriate functions for the eigenfunction expansion are spherical harmonics. Thus, we have, in analogy with (2),

$$\psi(\lambda, \phi) = \sum \sum_{|j|<k} \psi_k^j P_k^j(\sin\phi)\, e^{ij\lambda}, \tag{12}$$

where λ is longitude, ϕ is latitude and P_k^j the associated Legendre function. In the above expansion, k is the total wavenumber and j is the azimuthal wavenumber.

Consider a triad interaction among the components ψ_k^j, ψ_m^l and ψ_q^p. Without loss of generality, we may consider $k < m < q$. Such an interaction will be non-trivial if $j+l+p = 0$, $k+q > m$ and $k+m+q$ is odd. We shall make use of the first two selection rules, but ignore the third since we are interested in relative numbers of interactions. We may then form two quantities S and Q,

$$S = \frac{q(q+1) - m(m+1)}{m(m+1) - k(k+1)}, \tag{13}$$

$$Q = \frac{k(k+1)}{q(q+1)} S, \tag{14}$$

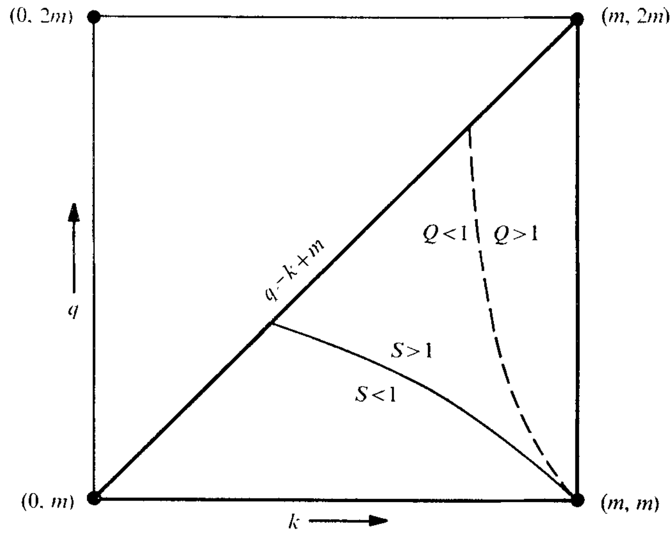

FIGURE 2. Characteristics of energy and enstrophy exchange in spherical co-ordinates. Every integer point within the triangle represents a possible interaction triad. Those points to the left of and below the solid curve imply $S < 1$; those points to the left of and below the dashed curve imply $Q < 1$. Because of the redundancy of interaction, the numbers of interactions are not directly proportional to the enclosed areas.

m	$S < 1$	$S > 1$	$Q < 1$	$Q > 1$
10	28·9	71·1	63·9	36·1
30	29·0	71·0	61·0	39·0
50	29·2	70·8	60·4	39·6
70	29·1	70·9	59·8	40·2
100	29·1	70·9	59·6	40·4

TABLE 1. Relative numbers (%) of interactions obtained by counting

such that if $S > 1$ ($Q > 1$) more energy (enstrophy) flows to or from longer wavelengths and vice versa for $S < 1$ ($Q < 1$).

Again, we choose a particular value of m and consider all possible interactions consistent with the assumed ordering and the selection rules and ask what relative numbers of interactions lead to S or Q greater or less than one. In figure 2 we show the region of possible values of k and q. The curves $S = 1$ and $Q = 1$ each divide the region into two areas as indicated. In this case, however, the areas of the various regions do not directly indicate the number of distinct interactions because of the redundancy of interactions with different j, l and p. In fact, for any point in the q, k plane, the number of distinct interactions is of the order of $(2q+1)(2k+1)$.

If we consider the limit of large m, we can estimate the relative number of interactions in various regions by quadrature. However, because of the difficulty of specifying an exact functional form for the density of interactions (i.e. we can only say that the density is of the order of $(2q+1)(2k+1)$), the estimate will be somewhat in error. By quadrature we find that about 25 % of the interactions imply $S < 1$ and about 55 % imply $Q < 1$. In view of this uncertainty in the density function, we have also performed these calculations by actually counting interactions for a number of different values of m. These results are shown in table 1. We note that, as m grows larger, the relative numbers of interactions converge such that about 30 % imply $S < 1$ and 60 % imply $Q < 1$.

5. Conclusion

We have shown that energy and enstrophy in a two-dimensional non-divergent flow cascade both to lower and higher wavenumbers, but that the relative magnitudes of the cascades are not consistently greater or less than unity. On the other hand, the majority of interactions are such that more energy flows to and from smaller wavenumbers while more enstrophy flows to and from larger wavenumbers. This latter concept is central to the concept of two inertial subranges which has been advanced by Kraichnan (1967).

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